Quasicrystals and icosians

1993 J. Phys. A: Math. Gen. 26 2829

(http://iopscience.iop.org/0305-4470/26/12/022)

J. Phys. A: Math. Gen. 26 (1993) 2829–2853. Printed in the UK

Quasicrystals and icosians

R V Moody† and J Patera‡

† Department of Mathematics, University of Alberta, Edmonton, Alberta T6G 2G1, Canada
‡ Centre de recherches mathématiques, Université de Montréal, CP 6128–A, Montréal, Québec H3C 3J7, Canada

Received 1 December 1992

Abstract. A family of quasicrystals of dimensions 1, 2, 3, 4 governed by the root lattice E_8 is constructed. The use of the icosian ring, found in the quaternions with coefficients in $\mathbb{Q}(\sqrt{5})$, allows simultaneous interpretation of the construction both in physical space and as a result of the standard 'cut-and-projection' method in double dimension. Icosians are seen to provide a natural co-ordination scheme for these quasicrystals. Nested sequences of quasicrystals form systems whose symmetries are all derivable from inflational and reflective symmetries directly related to the arithmetic of the icosians. The use of Coxeter diagrams clarifies the amazing relationship of E_8 and quasicrystal symmetries and leads to the fundamental chain $E_8 \supset D_6 \supset A_4 \supset A_1 \times A_1$ that underlies five-fold symmetry in quasicrystals. Decomposition of quasicrystals into concentric shells and a counting formula for the cardinalities of these shells is discussed.

1. Introduction

The six independent vectors of the reciprocal space required in the analysis of diffraction patterns of three-dimensional quasicrystals displaying icosahedral symmetry are no longer a matter of contention among physicists. More contentious is whether or not this fact demands a hyperspace theory formulated in higher than three-dimensional spaces or perhaps some amalgamation of a three-dimensional direct space and six-dimensional reciprocal space.

In this article we develop a theory that allows the three- and six-dimensional worlds to live together simultaneously in the same space; it is only a matter of interpretation which of the two is being discussed.

Our work is based on the root lattice E_8 and the largest of the non-crystallographic Coxeter groups H_4, together with a ring of quaternions with coefficients in a quadratic extension of the rational numbers (the icosians). Inherently it is a four-/eight-dimensional picture with standard three- and two-dimensional quasicrystals living as subsystems inside it. It brings together various ideas found in the literature, notably the series of articles [1–4] of the Tübingen group analysing the projections from A_4 and D_6 root lattices, the work of [5] where the connection between the E_8 root lattice, icosians and quasicrystals was first made, and the recent article [6, 7] where the E_8 shelling problem is first exposed. An important revelation was the ingenious and visually appealing realization of the symmetry group H_4 inside the Weyl group of E_8 given in [8].

The role of the ring of icosians in the theory of non-crystallographic Weyl groups goes back to [9]. Its connection with E_8 is pointed out in [10]. There are several passing references to icosians in the quasicrystal literature, but it seems to us that their real significance has not so far been realized. They are inherently simultaneously both four- and

0305-4470/93/122829+25$07.50

eight-dimensional and offer a natural and concise coordination scheme for all icosahedral quasicrystals coming from the D_6 cut-and-project procedure. In addition they have a multiplicative and arithmetic structure that allow us to multiply and otherwise manipulate quasicrystals in a natural way. Apart from making the entire quasicrystal symmetry, both isometrical and inflational, completely transparent, we feel that there is still much to be learned about the meaning of these remarkable structures.

In section 2 notation is fixed and some preliminary facts are recalled. We introduce the quadratic extension $\mathbb{F} = \mathbb{Q}(\sqrt{5})$ of the rational numbers $\mathbb{Q}$ and the space of quaternions $\mathbb{H}_{\mathbb{F}}$ with coefficients in $\mathbb{F}$. Apart from the standard $\mathbb{F}$-valued norm on $\mathbb{H}_{\mathbb{F}}$, note the two positive-definite rational-valued norms also defined in $\mathbb{H}_{\mathbb{F}}$. These norms play a decisive role in the sequel.

In section 3 a 1–1 correspondence between the simple roots of E_8 and certain quaternions is fixed (figure 1). This is the first crucial technical step of the article. The correspondence singles out the 120 icosians and their 120 τ-multiples; $\tau = (1+\sqrt{5})/2$. The icosian ring, generated by these 240 elements, is to be the stage on which the quasicrystals are defined in section 5. The advantage of our formulation of the E_8–icosian correspondence is that it allows us to read off many important facts directly from the Coxeter diagram of E_8. In particular, the D_6, A_4, and several other quasicrystals are straightforward particular subcases of those of E_8.

The inflational symmetry T, as introduced in section 4, is the defining symmetry of quasicrystals. In $\mathbb{H}_{\mathbb{F}}$ it is simply multiplication by τ. In the E_8 root lattice it determines quasicrystal eigenspaces. This is the second crucial step in this article.

In section 5 the quasicrystals Σ^r are defined in $\mathbb{H}_{\mathbb{F}}$ and in the four-dimensional eigenspaces of T (third crucial step). The parameter r is the radius of the acceptance domain which is taken here to be a sphere of appropriate dimension. Since in this article we are not concerned with the problem of quasiperiodic tiling (a quasicrystal consists here of points—vertices of some tiling), acceptance domains of more complicated shapes would offer only a minor variation to our examples (see also remarks in sections 8 and 10). New are the composition rules for our quasicrystals. An example of an A_4-quasicrystal is shown in figure 3.

The Coxeter group H_4, described in section 6, is the most important finite group of our problem. Its generating reflections are read off the E_8 diagram on figure 1, as well as the generators of its important subgroups H_3, the binary icosahedral group, and H_2, the dihedral group of order 10 (fourth crucial step).

In section 7 we show that inflational and H_4 symmetry account for all symmetries of the quasicrystal system Σ^r.

In section 8 quasicrystal systems in general are defined. These consist of infinite hierarchies of partially ordered quasicrystals all sharing the same symmetry group. The E_8 subcases corresponding to D_6, A_6, and $2A_1$ quasicrystals are discussed as examples.

In section 9 we introduce the shelling problem. If we decompose the E_8 root lattice as a sequence of concentric spherical shells then we simultaneously 'shell' the quasicrystal Σ^r. The properties of these shells (in particular their cardinalities) constitute the shelling problem first formulated by Sadoc and Mosseri [6, 7]. We announce here an explicit formula for these cardinalities. This formula, obtained by Moody and Weiss [11], corrects a conjectured formula given in [6]. Of particular interest is the direct appearance of the arithmetic of the icosian ring.

Structural lattice properties of the icosian ring are described in the appendix.

2. Notation and mathematical preliminaries

Let $\mathbb{F} = \mathbb{Q} + \mathbb{Q}\sqrt{5}$ denote the extension of rational numbers $\mathbb{Q}$ by $\sqrt{5}$ with the standard automorphism

$$' : \mathbb{F} \longrightarrow \mathbb{F} \qquad (a + b\sqrt{5})' = a - b\sqrt{5}. \tag{2.1}$$

Introduce the notation $\tau = \frac{1}{2}(1+\sqrt{5})$ and $\sigma = \frac{1}{2}(1-\sqrt{5})$, and note the identities

$$\sigma + \tau = 1 \qquad \sigma\tau = -1 \tag{2.2}$$

and their consequences $\sigma^2 = 1 + \sigma$ and $\tau^2 = 1 + \tau$ which we use often. The ring $\mathbb{Z}[\tau] = \mathbb{Z} + \mathbb{Z}\tau$ is the ring of integers of $\mathbb{F}$.

Let $Q = Q(E_8)$ denote the root lattice of the simple Lie group E_8, and let $\Delta \subset Q$ be the set of roots of E_8. The usual symmetric bilinear form on Q with values in the integers $\mathbb{Z}$ is denoted by $(\cdot \mid \cdot)$ and assumed to be normalized so that $(\alpha \mid \alpha) = 2$ for every root of E_8. The root lattice Q together with $(\cdot \mid \cdot)$ can be considered as a lattice in Euclidean space $\mathbb{R}^8$. Then we can consider also

$$V = \mathbb{Q}\text{-span of } Q \tag{2.3a}$$

$$V_{\mathbb{F}} = \mathbb{F}\text{-span of } Q \tag{2.3b}$$

which are the eight-dimensional spaces in $\mathbb{R}^8$ generated by Δ over $\mathbb{Q}$ and $\mathbb{F}$ respectively. The bilinear form $(\cdot \mid \cdot)$ is $\mathbb{Q}$-valued on V and $\mathbb{F}$-valued on $V_{\mathbb{F}}$.

The standard quaternionic algebra over $\mathbb{R}$ is denoted by $\mathbb{H}$, with the conjugation written as overbar:

$$\bar{\ } : \mathbb{H} \to \mathbb{H} \qquad \overline{(a_1 + ia_2 + ja_3 + ka_4)} = a_1 - ia_2 - ja_3 - ka_4. \tag{2.4}$$

Quaternions $a_1 + ia_2 + ja_3 + ka_4$ will often be written as the 4-tuples (a_1, a_2, a_3, a_4). The elements of

$$\mathbb{H}^0 := \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$$

are the pure quaternions characterized by the identity $\overline{x} = -x$.

Inside $\mathbb{H}$ we find the $\mathbb{F}$- and $\mathbb{Q}$-quaternion algebras defined as

$$\mathbb{H}_{\mathbb{F}} := \mathbb{F} + \mathbb{F}i + \mathbb{F}j + \mathbb{F}k \qquad \mathbb{H}_{\mathbb{Q}} := \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k.$$

and the quaternion rings

$$\mathbb{H}_{\mathbb{Z}[\tau]} := \mathbb{Z}[\tau] + \mathbb{Z}[\tau]i + \mathbb{Z}[\tau]j + \mathbb{Z}[\tau]k$$

$$\mathbb{H}_{\mathbb{Z}} := \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{Z} \text{ or } a, b, c, d \in \tfrac{1}{2} + \mathbb{Z}\}$$

with basis $\frac{1}{2}(1 + i + j + k), i, j, k$ over $\mathbb{Z}$.

We extend the field automorphism $'$ on $\mathbb{F}$ to a $\mathbb{Q}$-linear automorphism on $\mathbb{H}_{\mathbb{F}}$ by

$$(a, b, c, d)' = (a', b', c', d'). \tag{2.5}$$

The standard symmetric bilinear form (inner product) on $\mathbb{H}$ is given by

$$x \cdot y = \tfrac{1}{2}\{x\bar{y} + y\bar{x}\}. \tag{2.6}$$

In terms of coordinates this is the standard dot product on $\mathbb{R}^4$. In particular, we have the *quaternionic norm* of x,

$$N(x) := |x|^2 := x \cdot x = x\bar{x} \tag{2.7}$$

whose values on $\mathbb{H}_{\mathbb{F}}$ (resp $\mathbb{H}_{\mathbb{Z}}$) are in $\mathbb{F}$ (resp $\mathbb{Z}$).

Next we introduce a second bilinear form, $(\,\cdot\,)_\tau$, on $\mathbb{H}_{\mathbb{F}}$ with values in $\mathbb{Q}$ by combining with the $\mathbb{Q}$-linear map $x \mapsto (x)_\tau$ from $\mathbb{F} \to \mathbb{Q}$ defined by $(a + b\tau)_\tau \equiv a$. Thus

$$(x \cdot y)_\tau = a \qquad \text{if} \qquad x \cdot y = a + \tau b. \tag{2.8}$$

It is called the *rational form* relative to τ. Similarly, one introduces the rational form relative to σ, replacing τ by σ in (2.8). Correspondingly one speaks of rational norms $(x \cdot x)_\tau$ and $(x \cdot x)_\sigma$.

Any element $x \in \mathbb{H}_{\mathbb{F}}$ can be written uniquely as $x = q_1 + \tau q_2$, where $q_1, q_2 \in \mathbb{H}_{\mathbb{Q}}$. Then we have

$$\begin{aligned}(q_1 + \tau q_2) \cdot (q_1 + \tau q_2) &= q_1\bar{q}_1 + q_1\tau\bar{q}_2 + \tau q_2\bar{q}_1 + \tau q_2\tau\bar{q}_2 \\ &= N(q_1) + \tau^2 N(q_2) + \tau(q_2\bar{q}_1 + q_1\bar{q}_2) \\ &= N(q_1) + N(q_2) + \tau(N(q_2) + q_2\bar{q}_1 + q_1\bar{q}_2).\end{aligned} \tag{2.9}$$

Consequently, we have

$$((q_1 + \tau q_2) \cdot (q_1 + \tau q_2))_\tau = N(q_1) + N(q_2) \tag{2.10}$$

which shows that the rational norm of (2.8) on $\mathbb{H}_{\mathbb{F}}$ is positive-definite. In the same way the rational norm $(x \cdot x)_\sigma$ is also positive-definite on $\mathbb{H}_{\mathbb{F}}$.

3. The icosian ring and the E_8 root lattice

The following 120 unit quaternions:

$$\begin{aligned}&(\pm 1, 0, 0, 0) && \text{and all permutations} \\ &\tfrac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1) && \\ &\tfrac{1}{2}(0, \pm 1, \pm\sigma, \pm\tau) && \text{and all even permutations}\end{aligned} \tag{3.1}$$

called *icosians*, form a finite group I, the *group of icosians*, under the standard quaternionic multiplication [12]. The group is isomorphic to the binary icosahedral group.

The *icosian ring*, denoted by $\mathbb{I}$, is the $\mathbb{Z}$-span of I. Note that

$$\tau = \tfrac{1}{2}(\tau, 0, \sigma, 1) + \tfrac{1}{2}(\tau, 0, -\sigma, -1) \in \mathbb{I} \tag{3.2}$$

and similarly $\sigma \in \mathbb{I}$. Clearly $\bar{\mathbb{I}} = \mathbb{I}$ but notice that $\mathbb{I}' \neq \mathbb{I}$ (see the appendix for more on this).

Figure 1. The mapping $\pi_\| : \Delta \to I \cup \tau I$ given in terms of the mapping of the simple roots of E_8. $a_1 = \frac{1}{2}(-\sigma, -\tau, 0, -1)$, $\tau a_1 = \frac{1}{2}(1, -\tau^2, 0, -\tau)$, $a_2 = \frac{1}{2}(0, -\sigma, -\tau, 1)$, $\tau a_2 = \frac{1}{2}(0, 1, -\tau^2, \tau)$, $a_3 = \frac{1}{2}(0, 1, -\sigma, -\tau)$, $\tau a_3 = \frac{1}{2}(0, \tau, 1, -\tau^2)$, $a_4 = \frac{1}{2}(0, -1, -\sigma, \tau)$, $\tau a_4 = \frac{1}{2}(0, -\tau, 1, \tau^2)$.

There is a $\mathbb{Q}$-linear isometric isomorphism $\pi_\| : V \simeq \mathbb{H}_\mathbb{F}$ with $(\cdot \mid \cdot)$ used on V and $2(\,\cdot\,)_\tau$ used on $\mathbb{H}_\mathbb{F}$. Under the isomorphism, 240 roots of E_8 are mapped into the 120 icosians and their τ-multiples. We have

$$\pi_\| : \quad \Delta \to I \cup \tau I \qquad (\alpha \mid \beta) = 2(\pi_\|(\alpha) \cdot \pi_\|(\beta))_\tau\,. \tag{3.3}$$

The explicit mapping $\pi_\|$ set up in figure 1 realizes the isomorphism. For the rest of the paper we fix the notation

$$a_1 = \pi_\|(\alpha_1) \qquad a_2 = \pi_\|(\alpha_2) \qquad a_3 = \pi_\|(\alpha_3) \qquad \tau a_4 = \pi_\|(\alpha_4)$$

used in figure 1.

As an example, let us verify the following property of E_8 roots:

$$-1 = (\alpha_5 \mid \alpha_8) = 2(\pi_\|(\alpha_5) \cdot \pi_\|(\alpha_8))_\tau.$$

Using a_3 and a_4 from figure 1 we have

$$\begin{aligned} 2(\pi_\|(\alpha_5) \cdot \pi_\|(\alpha_8))_\tau &= (\tau a_3 \cdot a_4)_\tau = \tfrac{1}{2}(\tau(0, 1, -\sigma, -\tau) \cdot (0, -1, -\sigma, \tau))_\tau \\ &= \tfrac{1}{2}(\tau(-1 + \sigma^2 - \tau^2))_\tau = -1\,. \end{aligned}$$

Another isometric isomorphism $\pi_\perp : V \simeq \mathbb{H}_\mathbb{F}$ is built by replacing τ by σ in the definitions above. More precisely, one has

$$\begin{aligned} &\pi_\perp(x) = (\pi_\|(x))' \qquad \text{for all } x \in V \\ &\pi_\perp : \quad \Delta \to I' \cup \sigma I' \qquad (\alpha|\alpha) = 2(\pi_\perp(\alpha) \cdot \pi_\perp(\alpha))_\sigma\,. \end{aligned} \tag{3.4}$$

Thus if $(\pi_\|(x) \cdot \pi_\|(y)) = a + \tau b$, then $(\pi_\perp(x) \cdot \pi_\perp(y)) = a + \sigma b$.

Often it is practical to choose in V the basis of simple roots, writing α_i as the column matrix

$$\alpha_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T \tag{3.5}$$

where 1 is in the ith place. Relative to such a basis, the mappings $\pi_\|$ and $\pi_\perp$ are given by the 8×4 matrices

$$\pi_\| = \frac{1}{2}\begin{pmatrix} -\sigma & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\tau & -\sigma & 1 & -\tau & \tau & 1 & -\tau^2 & -1 \\ 0 & -\tau & -\sigma & 1 & 1 & -\tau^2 & 0 & -\sigma \\ -1 & 1 & -\tau & \tau^2 & -\tau^2 & \tau & -\tau & \tau \end{pmatrix} \tag{3.6}$$

$$\pi_\perp = \frac{1}{2}\begin{pmatrix} -\tau & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\sigma & -\tau & 1 & -\sigma & \sigma & 1 & -\sigma^2 & -1 \\ 0 & -\sigma & -\tau & 1 & 1 & -\sigma^2 & 0 & -\tau \\ -1 & 1 & -\sigma & \sigma^2 & -\sigma^2 & \sigma & -\sigma & \sigma \end{pmatrix}. \tag{3.7}$$

4. Inflation

The bijective mappings $\pi_\parallel$ and $\pi_\perp$ are essential ingredients of defining quasicrystals. Here we describe some of their properties.

The mapping $\pi_\parallel$ is determined by mapping four simple root vectors into certain icosians and the other four simple roots into τ-multiples of the same icosians (cf figure 1). The rational forms $(x \cdot y)_\tau$ and $(x \cdot y)_\sigma$ guarantee that the required angles and lengths of E_8 roots remain the same in $\mathbb{H}_\mathbb{F}$ as they are in V.

Now we want to construct a $\mathbb{Q}$-linear map $T : V \to V$ which mimics in V the multiplication of icosians by τ, namely

$$\pi_\parallel(Tx) = \tau\pi_\parallel(x) \qquad \text{for every } x \in V\,. \tag{4.1}$$

We get T by setting

$$\begin{aligned} T\alpha_1 &= \alpha_7 & T\alpha_7 &= \alpha_1 + \alpha_7 \\ T\alpha_2 &= \alpha_6 & T\alpha_6 &= \alpha_2 + \alpha_6 \\ T\alpha_3 &= \alpha_5 & T\alpha_5 &= \alpha_3 + \alpha_5 \\ T\alpha_8 &= \alpha_4 & T\alpha_4 &= \alpha_8 + \alpha_4\,. \end{aligned} \tag{4.2}$$

Obviously from (4.2) we have $T^2 - T - 1 = 0$ and T has two distinct eigenvalues, σ and τ, each occuring four times. We note also that if $x \in V$ and $\tilde{x} := \pi_\parallel(x)$ satisfies $N(\tilde{x}) \in \mathbb{Q}$, then we have

$$(x \mid Tx) = 2(\tilde{x} \cdot \tau\tilde{x})_\tau = 2(\tau N(\tilde{x}))_\tau = 0\,. \tag{4.3}$$

T is called the *inflation map* on V relative to $\pi_\parallel$.

Let us now extend $T, \pi_\parallel, \pi_\perp$ by $\mathbb{F}$-linearity to $V_\mathbb{F}$. Thus we have

$$\tilde{T} : \quad V_\mathbb{F} \to V_\mathbb{F} \qquad \tilde{\pi}_\parallel : \quad V_\mathbb{F} \to \mathbb{H}_\mathbb{F} \qquad \tilde{\pi}_\perp : \quad V_\mathbb{F} \to \mathbb{H}_\mathbb{F} \tag{4.4}$$

as well as the extension of $(\cdot \mid \cdot)$ to $V_\mathbb{F} \times V_\mathbb{F}$ by $\mathbb{F}$-bilinearity. The space $V_\mathbb{F}$ splits as

$$V_\mathbb{F} = (\tilde{T} - \sigma)V_\mathbb{F} \oplus (\tilde{T} - \tau)V_\mathbb{F} \tag{4.5}$$

into the direct sum of four-dimensional eigenspaces of $\tilde{T}$.

The map $\pi_\parallel : V \to \mathbb{H}_\mathbb{F}$ is 1–1 (a $\mathbb{Q}$-linear isomorphism). However, when we enlarge V to $V_\mathbb{F}$, the maps $\tilde{\pi}_\parallel$ and $\tilde{\pi}_\perp$ get non-trivial kernels

$$V_\sigma := \ker\tilde{\pi}_\parallel = \{(\tilde{T} - \tau)(x) \mid x \in V_\mathbb{F}\} = \{(T - \tau)(x) \mid x \in V\} \tag{4.6a}$$

$$V_\tau := \ker\tilde{\pi}_\perp = \{(\tilde{T} - \sigma)(x) \mid x \in V_\mathbb{F}\} = \{(T - \sigma)(x) \mid x \in V\} \tag{4.6b}$$

which are the eigenspaces σ and τ of $\tilde{T}$ of dimensions 4 over $\mathbb{F}$. The second equalities in (4.6) can be seen by observing that all the sets in question have $\mathbb{Q}$-dimensions equal to 8.

Let us prove that under $(\cdot \mid \cdot)$ one has

$$\ker\tilde{\pi}_\parallel \perp \ker\tilde{\pi}_\perp\,. \tag{4.7}$$

We have for all $x, y \in V$

$$(Tx - \tau x \mid Ty - \sigma y) = (Tx \mid Ty) - \tau(x \mid Ty) - \sigma(Tx \mid y) + \sigma\tau(x \mid y). \tag{4.8}$$

Since $Tx, Ty, x, y \in V$, we can work out these scalar products using (3.3):

$$(Tx \mid Ty) = 2\big(\tau\tilde{\pi}_{\|}(x) \cdot \tau\tilde{\pi}_{\|}(y)\big)_\tau = 2\big(\tau^2(\tilde{x} \cdot \tilde{y})\big)_\tau$$

where $\tilde{x} := \pi_{\|}(x)$ and $\tilde{y} := \pi_{\|}(y)$,

$$(x \mid \tilde{T}y) = 2(\tilde{x} \cdot \tau\tilde{y})_\tau = 2\Big(\tau(\tilde{x} \cdot \tilde{y})\Big)_\tau \qquad (\tilde{T}x \mid y) = 2(\tau\tilde{x} \cdot \tilde{y})_\tau = 2\Big(\tau(\tilde{x} \cdot \tilde{y})\Big)_\tau.$$

Thus (4.8) becomes

$$2\big\{\big(\tau^2(\tilde{x} \cdot \tilde{y})\big)_\tau - \tau\big(\tau(\tilde{x} \cdot \tilde{y}\big)_\tau - \sigma\big(\tau(\tilde{x} \cdot \tilde{y})\big)_\tau - (\tilde{x} \cdot \tilde{y})_\tau\big\}. \tag{4.9}$$

From $\sigma + \tau = 1$ and the linearity of the map $a + b\tau \longmapsto a$, we have

$$2\big\{\big(\tau^2(\tilde{x} \cdot \tilde{y}) - \tau(\tilde{x} \cdot \tilde{y}) - (\tilde{x} \cdot \tilde{y})\big)_\tau\big\} = 0 \tag{4.10}$$

due to $\tau^2 - \tau - 1 = 0$.

The maps $\tilde{\pi}_{\|}$ and $\tilde{\pi}_{\perp}$ look like orthogonal projections. However, $\mathbb{H}_{\mathbb{F}}$ is not a subspace of V or $V_{\mathbb{F}}$. In order to 'see' $\tilde{\pi}_{\|}$ as a projection, we do the following.

For any $x \in V$, we have the eigenspace decomposition

$$x = x_\tau + x_\sigma \qquad \text{where} \qquad x_\tau = \frac{Tx - \sigma x}{\sqrt{5}} \qquad x_\sigma = \frac{\tau x - Tx}{\sqrt{5}}. \tag{4.11}$$

Observe that

$$\tilde{\pi}_{\|}(x_\sigma) = \tilde{\pi}_{\|}\left(\frac{\tau x - Tx}{\sqrt{5}}\right) = 0 \qquad \text{and hence} \qquad \tilde{\pi}_{\|}(x_\tau) = \tilde{x}. \tag{4.12}$$

Thus $\tilde{\pi}_{\|}$ projects $x_\tau \longmapsto \tilde{x}$ and $x_\sigma \longmapsto 0$, and in the same way $\tilde{\pi}_{\perp} : x_\sigma \longmapsto \tilde{\pi}_{\perp}(x)$ and $x_\tau \longmapsto 0$.

The length of the image of a vector of V under $\pi_{\|}$ is, up to an overall scaling, the length of the component that projects onto it. In fact

$$(x_\tau \mid x_\tau) = \left(\frac{Tx - \sigma x}{\sqrt{5}} \,\middle|\, \frac{Tx - \sigma x}{\sqrt{5}}\right) = \tfrac{1}{5}\{(Tx \mid Tx) - 2(\sigma x \mid Tx) + (\sigma x \mid \sigma x)\}$$

where

$$(Tx \mid Tx) = 2(\tau^2(\tilde{x} \cdot \tilde{x}))_\tau$$

$$2(\sigma x \mid Tx) = 2\sigma(x \mid Tx) = 4\sigma(\tilde{x} \cdot \tau\tilde{x})_\tau = 4\sigma(\tau(\tilde{x} \cdot \tilde{x}))_\tau$$

$$(\sigma x \mid \sigma x) = \sigma^2(x \mid x) = 2\sigma^2(\tilde{x} \cdot \tilde{x})_\tau.$$

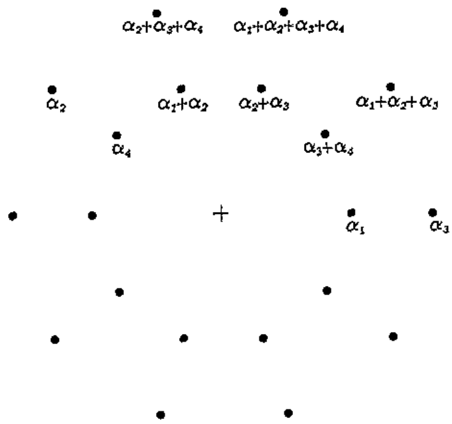


Figure 2. The $\pi_\parallel$-image of the roots of A_4. The simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of A_4 are respectively the simple roots $\alpha_3, \alpha_4, \alpha_5, \alpha_8$ of E_8 of figure 1 as mapped by $\pi_\parallel$.

Writing $(\tilde{x} \cdot \tilde{x}) = a + \tau b$, where $a, b \in \mathbb{Q}$, we get

$$\begin{aligned}(x_\tau \mid x_\tau) &= \tfrac{2}{5}\{(\tau^2(a+\tau b))_\tau - 2\sigma(\tau(a+\tau b))_\tau + \sigma^2(a+b\tau)_\tau\} \\ &= \tfrac{2}{5}\{((1+\tau)a + (1+2\tau)b)_\tau - 2\sigma(\tau a + (a+\tau)b)_\tau \\ &\qquad + (\sigma+1)(a+b\tau)_\tau\} \\ &= \tfrac{2}{5}\{(2+\sigma)a + (1-2\sigma)b\} = \tfrac{2}{5}(1+\sigma^2)\{a+\tau b\}.\end{aligned}$$

Thus

$$(x_\tau \mid x_\tau) = c(\pi_\parallel(x) \cdot \pi_\parallel(x)) \tag{4.13}$$

and in precisely the same way we find

$$(x_\sigma \mid x_\sigma) = c'(\pi_\perp(x) \cdot \pi_\perp(x)) \tag{4.14}$$

where the scaling constants in (4.13) and (4.14) are given by

$$c = \tfrac{2}{5}(2+\sigma) \qquad c' = \tfrac{2}{5}(2+\tau). \tag{4.15}$$

Now consider the orthogonal projections

$$p_\tau : \quad V_\mathbb{F} \longrightarrow (V_\mathbb{F})_\tau = V_\tau \tag{4.16a}$$

$$p_\sigma : \quad V_\mathbb{F} \longrightarrow (V_\mathbb{F})_\sigma = V_\sigma. \tag{4.16b}$$

Then we have the commutative diagrams

$$\begin{array}{ccc} V & \xrightarrow{\pi_\parallel} & \mathbb{H}_\mathbb{F} \\ & p_\tau\searrow \quad \nearrow\tilde{\pi}_\parallel & \\ & V_\tau & \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\pi_\perp} & \mathbb{H}_\mathbb{F} \\ & p_\sigma\searrow \quad \nearrow\tilde{\pi}_\perp & \\ & V_\sigma & \end{array} \tag{4.17}$$

and $\tilde{\pi}_\parallel$ and $\tilde{\pi}_\perp$ are dilations with scaling factors c and c' respectively. In this way $\pi_\parallel$ and $\pi_\perp$ mimic the projections p_τ and p_σ.

In the light of this discussion in we make an essential change in our viewpoint. Identify (as $\mathbb{Q}$-spaces) V and $\mathbb{H}_\mathbb{F}$ via the isometry $\pi_\parallel$. Then as soon as we view $x \in V = \mathbb{H}_\mathbb{F}$ as being in the four-dimensional $\mathbb{F}$-space we are looking at its projection by $\pi_\parallel(x)$, while x' is its projection $\pi_\perp(x)$ (see (3.4)).

An example where the result of a $\pi_\parallel$-mapping can be shown in two dimensions is found in figure 2.

Thus the two projections that we will use as the basis of the E_8 cut and project scheme of forming quasicrystals are $\pi_\parallel$ and $\pi_\perp$. In view of what we have said, they have an extremely simple and natural icosian interpretation. The scaling constants (4.15) that appear in relating this new picture to the usual one, which would involve p_τ and p_σ, amount to rescaling of the acceptance domain and the projected image, and are quite inessential.

5. Quasicrystals

For each $r > 0$ we define the quasicrystals $\Sigma^r \in \mathbb{I}$ and $\hat{\Sigma}^r \in \mathbb{I}$ as

$$\Sigma^r := \{x \in \mathbb{I} \mid N(x') < r^2\} = \{x \in \mathbb{I} \mid |x'| < r\} \tag{5.1a}$$

$$\hat{\Sigma}^r := \{x \in \mathbb{I} \mid |x'| \leqslant r\}. \tag{5.1b}$$

Let us point out several properties of the definition. It builds a four-dimensional quasicrystal entirely in 4-space and without reference to eight-space or the E_8 root lattice. Restricting the choice of x in the definition (5.1) to pure quaternions from $\mathbb{I}$, we get a three-dimensional quasicrystal Σ^{0r} which again is built without reference to the underlying sublattice D_6 (cf figure 1) of the E_8 root lattice. The process of changing dimensions is now relegated to the pair of fields $\mathbb{F}$ and $\mathbb{Q}$, the crucial ingredients being the involution (2.1) and the quaternionic norm (2.7).

It is immediately apparent from the definition that for all $r, s > 0$

$$\Sigma^r\Sigma^s \subset \Sigma^{rs} \tag{5.2}$$

$$\Sigma^r + \Sigma^s \subset \Sigma^{r+s} \tag{5.3}$$

$$\Sigma^r \subset \Sigma^s \quad \text{if } r < s\,. \tag{5.4}$$

Furthermore

$$\bigcup_{r>0} \Sigma^r = \mathbb{I}\,. \tag{5.5}$$

All these quasicrystals in (5.2)–(5.5) can be replaced by their closures $\hat{\Sigma}^r \in \mathbb{I}$.

An important aspect of this definition is the fact that Σ^r is clearly invariant under the group H of 14 400 symmetries $v \mapsto svt^{-1}$, $v \mapsto s\bar{v}t^{-1}$, $s, t \in I$ (see (7.1) and section 6).

The quasicrystal Σ^r can be equivalently formulated as

$$\Sigma^r := \{\pi_\parallel(x) \mid x \in Q,\ \pi_\perp(x)\cdot\pi_\perp(x) < r^2\} \qquad r > 0\,. \tag{5.6}$$

If we pull everything back to the V side we may define

$$\Sigma_Q^r := \{x_\tau \mid x \in Q, (x_\sigma \mid x_\sigma) < c'r^2\} \subset V_\tau \tag{5.7}$$

using (4.15) and then by (4.12)

$$\tilde{\pi}_\parallel : \quad \Sigma_Q^r \longrightarrow \Sigma^r\,. \tag{5.8}$$

In this form Σ^r is seen directly to be formed by the cut-and-projection method when the acceptance domain is taken as a sphere of radius $c'r^2$. Similar remarks apply to $\hat{\Sigma}^r$.

An example of a two-dimensional quasicrystal obtained using (5.6) is shown in figure 3. Note that figure 2 and figure 3 were obtained running the same computer program [13] for different values of r. More about A_4 quasicrystals and the choice of an acceptance domain is found in section 8.

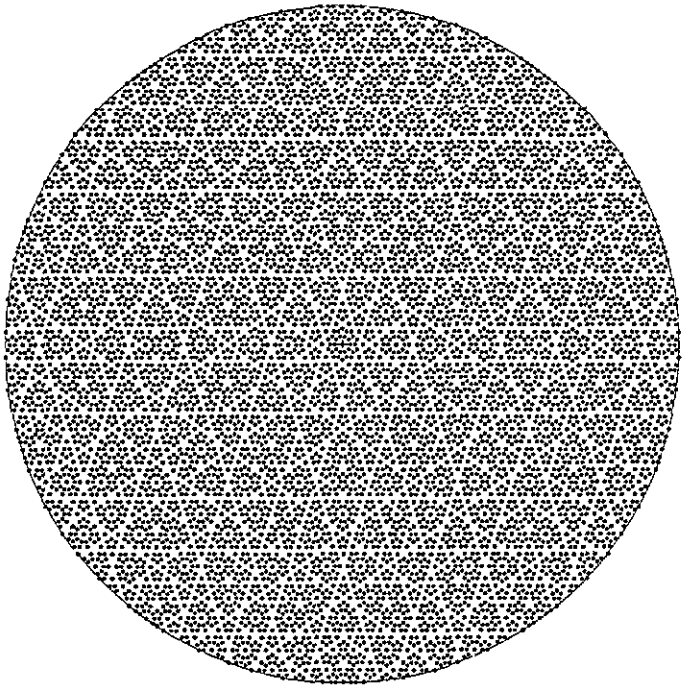


Figure 3. A circular window view of a planar quasicrystal Σ^r showing pentagonal symmetry. The projection $\pi_\parallel$ of figure 2 is applied to the points x of the A_4-root sublattice of the E_8 root lattice provided one has $\pi_\perp(x)\cdot\pi_\perp(x) < 25$.

Proposition 1. Let $r > 0$.

(i) Σ^r and Σ^r_Q are infinite.

(ii) Σ^r generates $\mathbb{I}$ as a $\mathbb{Z}$-module, Σ^r_Q generates V (as a $\mathbb{Z}$-module).

(iii) There exists a positive integer M such that for any $v \in V_\tau$ there exists $\{x_i \mid i = 1,\dots,M\} \subset \Sigma^r_Q$ so that v lies in the convex hull of the $\{x_i\}$.

Proof. Taking advantage of $\pi_\parallel$ we prove the results only for Σ^r. We begin by observing that if $x \in \Sigma^r$ then $|(\tau x)'| = |\sigma||x'| \leqslant |\sigma|r$ and so $\tau x \in \Sigma^{\tau^{-1}r} \subset \Sigma^r$. In the same way $\tau^{-1}x \in \Sigma^{\tau r}$. It follows at once that each set Σ^r is non-empty, and then infinite, proving (i).

Fix $r > 0$. By (5.5) for some $s > 0$, Σ^s contains a set of generators of $\mathbb{I}$ as a $\mathbb{Z}$-module. Also for some positive integer k, $\tau^k\Sigma^s \subset \Sigma^r$. Since $\tau^k\mathbb{I} = \mathbb{I}$ we see then that Σ^r contains a set of generators of $\mathbb{I}$, proving (ii).

In section 6 we will see that there is a reflection group H of order 14 400 acting irreducibly on $\mathbb{H}$ that stabilizes $\mathbb{I}$ and each quasicrystal Σ^r. Thus if $x \in \Sigma^r \setminus \{0\}$ then H_4x is a set of at most 14 400 points in Σ^r that spans $\mathbb{I}$ and forms the set of vertices of a convex polyhedron P_x inscribed in the sphere of radius $|x|$. Now clearly

$$\bigcup_{k=1}^{\infty} P_{\tau^k x} = \mathbb{H} \tag{5.9}$$

and (iii) follows. □

The origin of the ring $\mathbb{I}$ is always a point of the quasicrystal Σ^r. The origin is the centre of symmetry of Σ^r (cf the example in figure 3) and is also the centre of the acceptance domain defined by the requirement $N(x') < r^2$.

Let us fix a vector $\Phi \in V = \mathbb{H}_{\mathbb{F}}$, and let us call it a *phason*, and consider its projections Φ and Φ'. The quasicrystal Σ^r_Φ is defined by

$$\Sigma^r_\Phi := \{x + \Phi \mid x \in \mathbb{I},\ N((x+\Phi)') < r^2\}. \tag{5.10}$$

The quasicrystal Σ^r of (5.1) is obtained as the special case $\Phi = 0$. In general the origin of $\mathbb{I}$ does not belong to Σ^r_Φ and Σ^r_Φ has no centre of symmetry.

The phason family of quasicrystals containing Σ^r_Φ has fixed r and Φ taking values from the proximity cell (or the Voronoi domain or Wigner–Seitz cell) [14] of the root lattice Q around the origin of V.

6. The Coxeter group H_4

In this section we study certain subgroups of the Weyl group of E_8 which are pertinent to the quasiperiodic structures in two, three, and four dimensions.

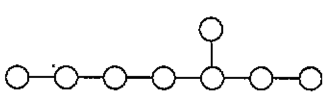
The Weyl group of E_8 is generated by the reflections $r_1, \ldots, r_8$ in planes orthogonal to the simple roots $\alpha_1, \ldots, \alpha_8$. This leads to the description in terms of the identities

$$W(E_8) = \langle r_1, r_2, \ldots, r_8 \mid (r_i r_j)^{m_{ij}} = 1 \rangle \tag{6.1a}$$

where m_{ij} is the matrix element of M given by

$$M = (m_{ij}) = \begin{pmatrix} 1 & 3 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 1 & 3 & 2 & 2 & 2 & 2 & 2 \\ 2 & 3 & 1 & 3 & 2 & 2 & 2 & 2 \\ 2 & 2 & 3 & 1 & 3 & 2 & 2 & 2 \\ 2 & 2 & 2 & 3 & 1 & 3 & 2 & 3 \\ 2 & 2 & 2 & 2 & 3 & 1 & 3 & 2 \\ 2 & 2 & 2 & 2 & 2 & 3 & 1 & 2 \\ 2 & 2 & 2 & 2 & 3 & 2 & 2 & 1 \end{pmatrix} \tag{6.1b}$$

or more succintly in terms of the Coxeter diagram



(6.2)

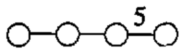
The Coxeter group H_4, of order 14 400, is defined abstractly by the presentation

$$H_4 = \{R_1, R_2, R_3, R_4 \mid (R_i R_j)^{m_{ij}} = 1\} \tag{6.3a}$$

where now

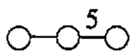
$$M = (m_{ij}) = \begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 3 & 2 \\ 2 & 3 & 1 & 5 \\ 2 & 2 & 5 & 1 \end{pmatrix} \tag{6.3b}$$

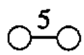
or equivalently by the Coxeter diagram



(6.4)

Of importance here also are the obvious subgroups H_3 and H_2 of H_4 whose diagrams are





(6.5)

respectively. The orders of these groups are 120 and 10 respectively.

Let $x \in \mathbb{H}$ with $N(x) = 1$. Then the Euclidean reflection in x (in the space $\mathbb{H} \simeq \mathbb{R}^4$ with the norm (2.7) is

$$R_x : v \to v - \frac{2v \cdot x}{x \cdot x} x = -x\bar{v}x\,. \tag{6.6}$$

Now let $x \in \Delta$ be a root which, when viewed in $\mathbb{H}_{\mathbb{F}}$, lies in I. Then $Tx \in \Delta$ identifies with $\tau x \in \tau I$ and $(Tx \mid x) = 0$, see (4.3). Let r_x and r_{Tx} be the reflections in $V = \mathbb{H}_{\mathbb{F}}$ with respect to $(\cdot \mid \cdot)$. Let us see that we have

$$r_{Tx} r_x = R_x \tag{6.7}$$

in our identification of V and $\mathbb{H}_{\mathbb{F}}$ under $\pi_{\parallel}$. Indeed,

$$\begin{aligned} r_{Tx} r_x v &= r_{Tx}(v - (v \mid x)x) = v - (v \mid x)x - (v \mid Tx)Tx \\ &= v - 2(v \cdot x)_\tau x - 2(v \cdot \tau x)_\tau \tau x\,. \end{aligned}$$

If we put $v \cdot x = p + q\tau$, where $p, q \in \mathbb{Q}$, then $(v \cdot x)_\tau = p$, $(v \cdot \tau x)_\tau = q$, and

$$r_{Tx} r_x v = v - 2(p + \tau q)x = v - 2(v \cdot x)x = R_x v\,. \tag{6.8}$$

Thus for all $x \in I$ we have (6.7).

Set

$$H := \langle R_1, R_2, R_3, R_4 \rangle \tag{6.9}$$

where

$$R_1 := r_1 r_7 \qquad R_2 := r_2 r_6 \qquad R_3 := r_3 r_5 \qquad R_4 := r_4 r_8\,. \tag{6.10}$$

In view of (6.7) these are reflections in $\mathbb{H}_{\mathbb{F}}$ and it is easy to see that they satisfy the Coxeter relations (6.3). In fact these relations are obvious from the standard Coxeter relations (6.1) of the Weyl group of E_8. The only unusual one (in fact the key one!) is

$$(R_3 R_4)^5 = (r_3 r_5 r_4 r_8)^5 = 1 \tag{6.11}$$

since $r_3 r_5 r_4 r_8$ is a Coxeter element for the underlying Weyl group of the Lie group of type A_4 and hence has order 5.

Alternatively we can work entirely inside $\mathbb{H}$. The reflections R_1, R_2, R_3, R_4 are determined by the Gramm matrix G of the basis a_1, a_2, a_3, a_4 under the quaternion inner product (2.6). Using the fact that

$$\frac{\tau}{2} = \cos\frac{\pi}{5} \tag{6.12}$$

we see that

$$G = \left(-\cos\frac{\pi}{m_{ij}}\right) \tag{6.13}$$

where $M = (m_{ij})$ is given by (6.3). It follows by the well known result [15–17] that the reflective generators (6.10) of H satisfy the Coxeter relations of (6.3) and, indeed,

$$H \simeq H_4 . \tag{6.14}$$

Furthermore since the matrix G is indecomposable the real representation of H on $\mathbb{H}$ is irreducible. The group H leaves invariant the $\mathbb{F}$-space $\mathbb{H}_{\mathbb{F}}$ which accordingly affords an irreducible $\mathbb{F}$-representation.

We next consider the set of all $\mathbb{Z}$-endomorhisms ϕ of $\mathbb{H}_{\mathbb{F}}$, defined for all $s, t \in I \times I$ as follows:

$$\phi_{(s,t)} : v \mapsto svt^{-1} \tag{6.15}$$

$$\gamma : v \mapsto \bar{v} . \tag{6.16}$$

The endomorphisms $\phi_{(s,t)}$ generate a group isomorphic to $I \times I/\langle(-1, -1)\rangle$ which, together with γ generate a group H' of order $(120)^2$. Since each reflection R_x of (6.6) with $x \in I$ lies in H',

$$H' = H \simeq H_4 . \tag{6.17}$$

We also make note of the subgroup of H_4 consisting of the 120 mappings generated by

$$\gamma \qquad \text{and} \qquad \phi_{(s,s)} \qquad s \in I . \tag{6.18}$$

We observe that if $x \in I$ is a pure quaternion ($\bar{x} = -x$) then the reflection R_x of (6.6) takes the form

$$R_x : \quad v \mapsto x\bar{v}x^{-1} \tag{6.19}$$

and that this reflection stabilizes the space $\mathbb{H}^0$ of pure quaternions. In particular, the subgroup

$$\langle R_2, R_3, R_4 \rangle \simeq H_3 \tag{6.20}$$

stabilizes $\mathbb{H}^0$. Since $|H_3| = 120$, we obtain in this way the entire group (6.18).

We now have three pictures of H_4: the abstract version of (6.3), the subgroup H_4 of the Weyl group of E_8 given by (6.9), and the group generated by the endomorphisms (6.15) and (6.16) of $\mathbb{H}$. In general we will identify these three groups and denote them by H_4. When confusion is possible we will simply state which context we wish to view it in.

There is one final point. For each $x \in I$ the reflection R_x of (6.6) stabilizes I and so I is the non-crystallographic root system of type H_4. The set $\{a_1, a_2, a_3, a_4\}$ is a base for I in the sense of finite root system: each element of I is uniquely expressible as a non-negative or non-positive linear combinations of a_1, a_2, a_3, a_4 [16, 17].

Proposition 2. The group H_4 viewed as a subgroup of $W(E_8)$ is characterized by

$$H_4 = \{w \in W(E_8) \mid wT = Tw\}. \tag{6.21}$$

The spaces V_σ and V_τ afford irreducible representations of H_4.

Proof. H_4 is generated by the elements $R_1, \ldots, R_4$ of (6.10). These are reflections when viewed as endomorphisms of $\mathbb{H}$ over $\mathbb{F}$ and hence commute with T as endomorphisms of V.

Conversely suppose that $w \in W(E_8)$ commutes with T. Then w acts as an $\mathbb{F}$-linear endomorphism on $\mathbb{H}$ and $w(I \cup \tau I) = I \cup \tau I$. Now we claim that $wI = I$. Indeed, if $x \in I$ and $w(x) = \tau y$ for some $y \in I$ then $\tau x \in \tau I$ and $w(\tau x) = \tau^2 y = y + \tau y$. Since $(y + \tau y \mid y + \tau y) = 4$ we see that $y + \tau y \notin I \cup \tau I$. Thus $wI = I$.

If I_+ is a positive system for the root system I (see [16, 17]) then so is wI_+ and hence there is a $w' \in H_4$ with $w^{-1}w' : I_+ \to I_+$. Then $w^{-1}w'$ preserves the unique simple system determined by I_+. It is easy to see that $w^{-1}w'$ must effect a diagram automorphism and thus in our case of H_4, $w^{-1}w' = 1$, i.e. $w = w' \in H_4$.

The statements about V_σ and V_τ follow by using the isomorphisms (4.17). □

7. Symmetries of quasicrystals

The whole point of introducing H_4 is of course that each quasicrystal Σ^r is H_4-invariant. Using (6.15) and (6.16) we have

$$\begin{aligned} v \in \Sigma^r &\Rightarrow v \in \mathbb{I} \qquad |v'| < r \\ &\Rightarrow v \in \mathbb{I} \qquad |s'v't'^{-1}| < r \quad \text{for all} \quad s, t \in I \\ &\qquad \text{since} \quad |s'| = |s|' = 1 = |t'^{-1}| \\ &\Rightarrow \phi_{(s,t)}(v) \in \mathbb{I} \qquad |(\phi_{(s,t)}(v))'| < r \\ &\Rightarrow \phi_{(s,t)}(v) \in \Sigma^r . \end{aligned} \tag{7.1}$$

In the same way $v \in \Sigma^r \Rightarrow \bar{v} \in \Sigma^r$.

Considering the pure quasicrystals Σ^{0r} and using (6.19) and (6.20) we see that Σ^{0r} is H_3-invariant. Thus the D_6 subdiagram of E_8 obtained from figure 1 by deleting nodes α_1 and α_7 is responsible for the icosahedral quasicrystals. The role of D_6 in determining the icosahedral symmetry by the cut and projection method is due to [7–9]. However, our acceptance domain is a ball in 3-space rather than a polytope.

Quasicrystals Σ^r and Σ^s are said to be isomorphic if there exists a $\mathbb{Z}$-linear map $\phi : \mathbb{I} \to \mathbb{I}$ so that $\phi(\Sigma^r) = \Sigma^s$.

Proposition 3. Let $\phi : \mathbb{I} \to \mathbb{I}$ be a $\mathbb{Z}$-linear map and suppose that for some $r, s > 0$, $\phi(\Sigma^r) \subset \Sigma^s$. Then ϕ is a $\mathbb{Z}[\tau]$-linear map.

Proof. We have to prove that $\phi(\tau x) = \tau\phi(x)$ for all $x \in \mathbb{I}$. Using $\pi_\parallel$ we may pull everything back to Q and assume that $\phi : Q \to Q$ is a $\mathbb{Z}$-linear map with $\phi(\pi_\parallel^{-1}(\Sigma^r)) \subset \pi_\parallel^{-1}(\Sigma^s)$. Thus we have to prove that $\phi T = T\phi$.

We observe that ϕ lifts uniquely to a $\mathbb{Q}$-linear map on V and to an $\mathbb{F}$-linear map on $V_{\mathbb{F}}$. We will also denote these maps by ϕ. Consider the $\mathbb{Q}$-linear map

$$\lambda:\ V_\tau \longrightarrow V_\sigma \qquad v \mapsto (\phi v)_\sigma \qquad \text{for all } v \in V\,. \tag{7.2}$$

By assumption $\phi(\pi_\parallel^{-1}(\Sigma^r)) \subset \pi_\parallel^{-1}(\Sigma^s)$. Since for all $x \in Q$,

$$x \in \pi_\parallel^{-1}(\Sigma^r) \Longleftrightarrow (x_\sigma \mid x_\sigma) < c'r^2 \tag{7.3}$$

we have for all $x \in Q$

$$(x_\sigma \mid x_\sigma) < c'r^2 \Longrightarrow (\phi x)_\sigma \mid (\phi x)_\sigma < c's^2\,. \tag{7.4}$$

As x_τ runs over Σ_Q^r, the set $\{x_\sigma\}$ is bounded; hence also the sets $\{\phi(x_\sigma)\}$ and $\{\phi(x_\sigma)_\sigma\}$ are bounded. Since also the set $\{\phi(x)_\sigma\}$ is bounded we have from

$$\phi(x)_\sigma = \phi(x_\sigma)_\sigma + \phi(x_\tau)_\sigma \tag{7.5}$$

that the set $\{\phi(x_\tau)_\sigma\}$ is bounded; in short $\lambda(\Sigma_Q^r)$ is bounded.

According to the proposition 1, any point v of V_τ is expressible as a linear combination $\sum_{i=1}^M c_i x_i$ where $c_i \geqslant 0$, $\sum c_i = 1$, $x_i \in \Sigma_Q^r$ and M is independent of v. Then $\lambda(v) = \sum_{i=1}^M c_i \lambda(x_i)$ and we see that $\lambda(V_\tau)$ is bounded. Since λ is linear we obtain $\lambda = 0$. This proves that $(\phi V_\tau)_\sigma = 0$, i.e. $(\phi V_\tau) \subset V_\tau$.

Now we can define the linear map $f : V \to V$ by

$$\phi((T-\sigma)v) = (T-\sigma)f(v) \tag{7.6}$$

(since $T-\sigma$ is injective on V this is well defined). Then

$$\phi(Tv) - \sigma\phi v = T(fv) - \sigma f(v) \tag{7.8}$$

and using the independence of 1 and σ over $\mathbb{Q}$, we obtain

$$\phi T(v) = Tf(v) \qquad \phi(v) = f(v) \tag{7.9}$$

whence $\phi = f$ and $\phi T = T\phi$. □

Proposition 4. Let $\phi : \mathbb{I} \to \mathbb{I}$ be a $\mathbb{Z}$-linear map and suppose that for some $r, s > 0$, $\phi\Sigma^r = \Sigma^s$. Then ϕ is $\mathbb{Z}[\tau]$-linear and

(i) If $r = s$ then $\phi \in H_4$;

(ii) If $r \neq s$ then $s/r = \tau^k$ for some $k \in \mathbb{Z}$ and $\phi \in \tau^{-k}H_4$.

Conversely if $r, s > 0$ and $\phi : \mathbb{I} \to \mathbb{I}$ is a $\mathbb{Z}$-linear mapping satisfying (i) and (ii) then $\phi\Sigma^r = \Sigma^s$.

Proof. Since Σ^r and Σ^s contain bases for $\mathbb{I}$ (over $\mathbb{Z}$), ϕ is surjective and hence bijective. By (i), ϕ is τ-linear, i.e. ϕ is a $\mathbb{Z}[\tau]$-linear map. Extend ϕ to an $\mathbb{F}$-linear map of $\mathbb{H}$ onto itself. Let S^t be the ball of radius t, $t > 0$, in $\mathbb{H}_{\mathbb{F}}$.

Now

$$\Sigma^r = \{x \in \mathbb{I} \mid |x'| < r\} = \{y' \mid y \in \mathbb{I}',\ |y| < r\}\,.$$

Let $\psi : \mathbb{H}_{\mathbb{F}}$ be the $\mathbb{F}$-bilinear map $v \mapsto (\phi(v'))'$. Then we claim

$$\psi(S^r \cap \mathbb{I}') \subset S^s .$$

Indeed, $y \in S^r \cap \mathbb{I}' \Rightarrow y' \in \Sigma^r \Rightarrow \phi(y') \in \Sigma^s \Rightarrow (\phi(y'))' \in S^s$. Since $S^r \cap \mathbb{I}'$ is dense in S^r and ψ is linear we conclude that $\psi(S^r) \subset S^s$. Using ϕ^{-1} we conclude in the same way that $\psi^{-1}(S^s) \subset S^r$. Thus

$$\psi(S^r) = S^s. \tag{7.10}$$

It follows that ψ is a dilation:

$$\psi(x) \cdot \psi(y) = (s/r)^2 x \cdot y \qquad \forall\, x, y \in \mathbb{H}_{\mathbb{F}} . \tag{7.12}$$

Choosing $x, y \in \mathbb{I}'$ with $x \cdot y = 1$, we obtain (because $\psi(\mathbb{I}') \subset \mathbb{I}'$ and $\mathbb{I}' \cdot \mathbb{I}' = \mathbb{Z}[\tau]$), $(s/r)^2 \in \mathbb{Z}[\tau]$. In the same way from ψ^{-1} we obtain $(r/s)^2 \in \mathbb{Z}[\tau]$, therefore

$$(s/r)^2 \in \mathbb{Z}[\tau]^{\times}_{>0} = \{\tau^k \mid k \in \mathbb{Z}\} = \langle\tau\rangle . \tag{7.13}$$

Suppose now that $s = r$ so ψ is an isometry. Recall $\pi_\perp : Q \to \mathbb{I}'$. We use this to lift ψ back to a linear mapping $\hat{\psi}$ on Q that commutes with T. Also by (3.4)

$$(\hat{\psi}(x) \mid \hat{\psi}(y)) = 2(\psi\pi_\perp(x) \cdot \psi\pi_\perp(y))_\sigma = 2(\pi_\perp(x) \cdot \pi_\perp(y))_\sigma = (x \mid y) . \tag{7.14}$$

Thus $\hat{\psi} : Q \to Q$ preserves the bilinear form $(\cdot \mid \cdot)$. It follows that $\hat{\psi} \in W(E_8)$, i.e $\hat{\psi} \in \mathrm{Aut}\,(\Delta)$. Since also $\hat{\psi}T = T\hat{\psi}$, we have by the proposition 2

$$\hat{\psi} \in H_4 . \tag{7.15}$$

Consequently ψ is a mapping of the form $v \to svt^{-1}$ or $v \to s\bar{v}t^{-1}$, $s, t \in \mathbb{I}'$. Returning to ϕ,

$$\phi(v) = (\psi(v'))' = (sv't^{-1})' = s'vt'^{-1} \tag{7.16a}$$

or

$$\phi(v) = s'\bar{v}t'^{-1} . \tag{7.16b}$$

where $s', t' \in I$. Thus we have proved that $\phi \in H_4$, completing the proof of (i).

Now we consider the case $r \neq s$. Let $a = (s/t)^2$. If $\{e_1, e_2, e_3, e_4\}$ is a basis for $\mathbb{H}$ and $G = (g_{ij}) = (e_i \cdot e_j)$ is the corresponding Gramm matrix, then writing $\psi e_i = \sum a_{ji} e_j$ and $A = (a_{ji})$ we obtain

$$aG = A^T G A . \tag{7.17}$$

The Gramm matrix G is positive-definite since $(\,\cdot\,)$ is a positive-definite scalar product on $\mathbb{H}$. But $\{e_1', e_2', e_3', e_4'\}$ is also a basis for $\mathbb{H}$ and

$$(e_i' \cdot e_j') = \tfrac{1}{2}\{e_i'\bar{e}_j' + e_j'\bar{e}_i'\} = \tfrac{1}{2}\{e_i\bar{e}_j + e_j\bar{e}_i\}' = (e_i \cdot e_j)' = g_{ij}' . \tag{7.18}$$

Thus G' is also positive-definite. From (7.17)

$$a'G' = A'^T G' A' \tag{7.19}$$

and since $A'^T G' A'$ is positive-definite, $a' > 0$. But $a = \tau^m$ so $a' = \sigma^m > 0$ whence $m = 2k$ for some $k \in \mathbb{Z}$ and so $s/r = \tau^k$ for some $k \in \mathbb{Z}$.

Finally $\tau^k\phi : \mathbb{I} \to \mathbb{I}$ and the corresponding mapping $v \mapsto (\tau^k\phi(v'))'$ is an isometry. Hence we are in case (i) and $\tau^k\phi \in H_4$. □

Proposition 5. Let $\mathcal{X}$ be the set of equivalence classes of Σ under isomorphism. Then $\mathcal{X}$ is a group isomorphic to $\mathbb{R}_+/\langle\tau\rangle$ via the map $\Sigma^r \longrightarrow r\langle\tau\rangle$.

Proof. $\Sigma^r \simeq \Sigma^s \Leftrightarrow s = \tau^k r$ for some $k \in \mathbb{Z}$. The group structure on $\mathcal{X}$ is induced from (5.2). □

Note that

$$\mathbb{R}_+/\langle\tau\rangle \simeq \mathbb{R}/\mathbb{Z} \tag{7.20}$$

under the mapping

$$r \longmapsto \frac{\log r}{\log \tau} \qquad \bmod \mathbb{Z}. \tag{7.21}$$

8. The quasicrystal systems and subsystems

In our definition of quasicrystals in section 5 we have adopted as the acceptance domains the open balls centred at origin. Of course, any bounded neighbourhood of the origin invariant under the appropriate symmetry group G can be used to produce a quasicrystal with G-invariance. The choice of open balls is particularly convenient from an algebraic point of view, but from the point of view of tilings it has been very important to use various polytopes as acceptance domains [1–4]. The sheer enormity of the number of possible choices makes it difficult to make a coherent scheme out of all available quasicrystals. However, we may notice that if we choose a polytope P we can, by suitably scaling P, arrive at a system

$$\Omega := \{\Omega^r\}_{r>0} \tag{8.1}$$

of quasicrystals and that the system is 'commensurable' with Σ in the sense that for all $s > 0$ there are positive real numbers r_1, r_2, r_3, r_4 so that

$$\Sigma^{r_1} \subset \Omega^s \subset \Sigma^{r_2} \qquad \text{and} \qquad \Omega^{r_3} \subset \Sigma^s \subset \Omega^{r_4}. \tag{8.2}$$

This suggests that we introduce the notion of a *system of quasicrystals.*

Let $M = \sum_{i=1}^m \mathbb{Z}a_i$ be a finitely generated subgroup of $\mathbb{R}^n$ (in general $m > n$). Let G be a subgroup of $GL(N)$ that leaves M invariant. A subset Λ_0 of M is a G-invariant M-quasilattice of M if Λ_0 is G-invariant, discrete and uniform (uniform means that there is a real number $R > 0$ such that every ball of radius R in $\mathbb{R}^n$ intersects Λ_0 non-trivially). By a system of G-invariant M-quasilattices we mean a set $\{\Lambda_i\}_{i\in J}$ of G-invariant quasilattices of M together with the partial ordering of inclusion satisfying

(i) for all $i, j \in J$ there exist $k, l \in J$ so that $\Lambda_k \subset \Lambda_p \subset \Lambda_l$, $p = i, j$;
(ii) $\cup_{i\in J}\Lambda_i = M$, $\cap_{i\in J}\Lambda_i = (0)$.

A second system of G-invariant M-quasilattices $\Lambda' = \{\Lambda'_{i'} \mid i \in I'\}$ is *commensurable* with Λ if for all $i \in I$, $i' \in I'$ there exist $j'_1, j'_2 \in I'$, $j_1, j_2 \in I$ so that

$$\Lambda'_{j'_1} \subset \Lambda_i \subset \Lambda'_{j'_2} \qquad \Lambda_{j_1} \subset \Lambda'_i \subset \Lambda_{j_2}.$$

Thus $\Sigma = \{\Sigma^r \mid r > 0\}$ is a system of H_4-invariant $\mathbb{I}$-quasilattices and the choice of any acceptance domains rP, $r > 0$, where P is some H_4-invariant polytope centred at the origin leads to a commensurable system.

A *system of quasicrystals* is a system of quasilattices in which each Λ_i is a quasicrystal. Unfortunately at the present time a suitable general definition of quasicrystals is elusive. Generally, it is believed that it should at least involve a statement about the Fourier transform of the set of points involved. We do not have anything to add to this question.

A symmetry of a quasilattice system Λ is a $\mathbb{Z}$-linear mapping ϕ on M that induces a mapping $\bar{\phi}$ on I so that $\phi\Lambda_i = \Lambda_{\bar{\phi}(i)}$ for all $i \in I$. Thus we have proved in section 7 that for our quasicrystal system Σ the symmetries comprise precisely the group generated by H_4 and the inflations τ^k, $k \in \mathbb{Z}$. If ϕ is a symmetry of Λ and Λ' is commensurable with Λ then we can 'complete' Λ' to a commensurable quasilattice Λ'' containing Λ' that also has ϕ as a symmetry. In this way the symmetry group of a commensurable collection of systems of quasilattices is a well defined object.

In the rest of this section we discuss some specific subsystems of the E_8 system.

First we consider the A_4 quasicrystal system used for the examples in figures 2 and 3. For this we choose from figure 1 the subdiagram of the E_8 simple roots $\alpha_3, \alpha_4, \alpha_5, \alpha_8$ spanning the A_4 diagram and rename the roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ respectively according to the standard A_4 convention. Thus we have

$$\begin{aligned} \pi_\parallel(\alpha_1) &= a_3 = \tfrac{1}{2}(0, 1, -\sigma, -\tau) \qquad & \pi_\parallel(\alpha_2) &= \tau a_4 = \tfrac{\tau}{2}(0, -1, -\sigma, \tau) \\ \pi_\parallel(\alpha_3) &= \tau a_3 = \frac{\tau}{2}(0, 1, -\sigma, -\tau) \qquad & \pi_\parallel(\alpha_4) &= a_4 = \tfrac{1}{2}(0, -1, -\sigma, \tau) \end{aligned} \tag{8.3}$$

and also

$$\begin{aligned} \pi_\perp(\alpha_1) &= a'_3 = \tfrac{1}{2}(0, 1, -\tau, -\sigma) \qquad & \pi_\perp(\alpha_2) &= \sigma a'_4 = \frac{\sigma}{2}(0, -1, -\tau, \sigma) \\ \pi_\perp(\alpha_3) &= \sigma a'_3 = \frac{\sigma}{2}(0, 1, -\tau, -\sigma) \qquad & \pi_\perp(\alpha_4) &= a'_4 = \tfrac{1}{2}(0, -1, -\tau, \sigma)\,. \end{aligned} \tag{8.4}$$

The two-dimensional quasicrystal $\Sigma^r \cap (\mathbb{F}a_3 + \mathbb{F}a_4)$ is then built for a given $r > 0$ by repeating the following steps. One takes a point x in the A_4-root lattice,

$$x = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 + x_4\alpha_4 \qquad x_1, \ldots, x_4 \in \mathbb{Z} \tag{8.5}$$

and its projection $\pi_\perp(x)$,

$$\begin{aligned} \pi_\perp(x) &= x_1\pi_\perp(\alpha_1) + x_2\pi_\perp(\alpha_2) + x_3\pi_\perp(\alpha_3) + x_4\pi_\perp(\alpha_4) \\ &= (x_1 + \sigma x_3)a'_3 + (x_4 + \sigma x_2)a'_4\,. \end{aligned} \tag{8.6}$$

Then one selects the point

$$\pi_\parallel(x) = (x_1 + \tau x_3)a_3 + (x_4 + \tau x_2)a_4 \tag{8.7}$$

in the plane spanned by the unit quaternions a_3 and a_4 iff the quaternionic norm $N(\pi_\perp(x))$ verifies the inequality

$$N(\pi_\perp(x)) < r^2\,. \tag{8.8}$$

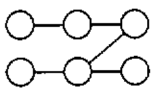
Note that the quaternionic nature of the basis vectors plays no role in the process of selection.

The symmetry group of the A_4 system of quasicrystals induced from the symmetries of the E_8 quasicrystal system is, according to (6.10), the group generated by the dihedral group

$$H_2 := \langle R_3, R_4 \rangle \qquad |H_2| = 10 \tag{8.9}$$

and the group of inflations $\langle \tau \rangle$.

Undoubtedly the most important subsystem of the E_8 quasicrystal system is given by the D_6 subdiagram



(8.10)

of the E_8 diagram. It properly contains the two-dimensional A_4 system discussed above. In the set-up of figure 1, the six-dimensional root lattice of D_6 is projected onto the three-dimensional space of pure quaternions of $\mathbb{I}$ as was already pointed out in section 5.

Renumbering the E_8 roots $\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_8$, which span the D_6 subdiagram, respectively as $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$, and choosing a value of $r > 0$, we build a three-dimensional quasicrystal Σ^{0r} point by point as before. Explicitly, a lattice point

$$x = x_1\alpha_1 + \cdots + x_6\alpha_6 \qquad x_1, \ldots, x_6 \in \mathbb{Z} \tag{8.11}$$

of the D_6 root lattice gives rise to the two projections:

$$\pi_\perp(x) = x_1\pi_\perp(\alpha_1) + \cdots + x_6\pi_\perp(\alpha_6) = (x_1 + \sigma x_5)a_2' + (x_2 + \sigma x_4)a_3' + (x_6 + \sigma x_3)a_4' \tag{8.12}$$

$$\pi_\parallel(x) = x_1\pi_\parallel(\alpha_1) + \cdots + x_6\pi_\parallel(\alpha_6) = (x_1 + \tau x_5)a_2 + (x_2 + \tau x_4)a_3 + (x_6 + \tau x_3)a_4 . \tag{8.13}$$

Whenever one has

$$N(\pi_\perp(x)) = \pi_\perp(x) \cdot \pi_\perp(x) < r^2 \tag{8.14}$$

the point (8.13) is selected in the 3-space in which the pure quaternions a_2, a_3, a_3 of figure 1 serve as basis vectors.

The induced symmetry group of the system of D_6 quasicrystals is generated by the icosahedral group $H_3 = \langle R_2, R_3, R_4 \rangle$ of (6.20) and the inflations $\langle \tau \rangle$.

There are three other possible ways to cut a pair of vertically aligned nodes of the E_8 diagram of figure 1. Each of them leads to a different subdiagram hence to a different qusicrystal system in 3-space. Here we specify each of the three cases by its symmetry group G and by the type of the subdiagram one gets. They are the following:

$$G := \langle R_1, R_3, R_4 \rangle \qquad |G| = 120 \qquad 2A_1 + A_4 \tag{8.15}$$

$$G := \langle R_1, R_2, R_4 \rangle \qquad |G| = 12 \qquad 2A_1 + 2A_2 \tag{8.16}$$

$$G := \langle R_1, R_2, R_3 \rangle \qquad |G| = 24 \qquad 2A_3 . \tag{8.17}$$

The quasicrystals in these cases are built in the same way as above using the appropriate quaternions from among a_1, a_2, a_3, a_4 for basis vectors of the 3-space.

Finally one may wish to see the best known of all quasicrystals: the one-dimensional one. Following our approach, we cut the E_8 diagram of figure 1, removing from it all but one pair (any pair) of vertically aligned nodes. We are left with the $A_1 + A_1$ diagram. Call the simple roots α and β. They project as

$$\pi_{\|}(\alpha) = a \qquad \pi_{\perp}(\alpha) = a' \qquad \pi_{\|}(\beta) = \tau a \qquad \pi_{\perp}(\beta) = \sigma a' \tag{8.18}$$

where a is the icosian left in the cut diagram. For a fixed r, every point $x = x_1\alpha + x_2\beta$, $x_1, x_2 \in \mathbb{Z}$, of the $A_1 + A_1$ lattice satisfying the inequality

$$N((x_1 + \sigma x_2)a') = (x_1 + \sigma x_2)^2 a'\bar{a}' = (x_1 + \sigma x_2)^2 < r^2$$

determines a point of the one-dimensional quasicrystal, namely the point

$$\pi_{\|}(x) = (x_1 + \tau x_2)a\,.$$

The basis vector a is irrelevant for the one-dimensional problem. Indeed, the construction really takes place in $\mathbb{Z}[\tau]$ and the system of quasicrystals is

$$\Lambda^r := \{x \in \mathbb{Z}[\tau] \mid |x'| < r\}\,. \tag{8.19}$$

It is interesting to observe that there is a series of relative quasicrystals based on $\mathbb{I}$ which arise by altering the acceptance domain to a spherical shell: for $0 < r < R$,

$$\Sigma^{r,R} = \{x \in \mathbb{I} \mid r < |r'| < R\} = \Sigma^R \backslash \hat{\Sigma}^r.$$

These sets retain the H_4-symmetry but lose the inflational symmetry.

9. Shelling quasicrystals

In [6] Sadoc and Mosseri introduce the notion of the shelling of quasicrystals and make a remarkable conjecture concerning the number of points on shells. In this section we describe the basic features of this process in the icosian setting.

The E_8 root lattice Q decomposes into a set of concentric shells

$$Q_N := \{x \in Q \mid (x \mid x) = 2N\} \qquad N = 0, 1, 2, \ldots\,. \tag{9.1}$$

This gives rise to the theta series

$$Q(q) = \sum_{x \in Q} q^{(x|x)} = \sum_{N=0}^{\infty} \operatorname{card}(Q_N) q^{2N} \tag{9.2}$$

for which there is the well known result [9]

$$\operatorname{card}(Q_N) = 240\left(\sum_{d|N} d^3\right). \tag{9.3}$$

We now introduce a shelling on each quasicrystal Σ^r: We set $\mathbb{I}_N := \pi_{\|}(Q_N)$ (in other words Q_N is viewed in $\mathbb{H}$) and

$$\Sigma^r_N := \Sigma^r \cap \mathbb{I}_N = \{x \in \mathbb{I}_N \mid |x'| < r\}. \tag{9.4}$$

For $x \in \mathbb{I}$ we have $x \in \Sigma_N^r$ if and only if

$$x \cdot x = N + m\tau \tag{9.5a}$$

$$0 \leqslant x' \cdot x' = N + m\sigma < r^2 . \tag{9.5b}$$

For some $m \in \mathbb{Z}$, from (9.5*b*) we have the equivalent condition

$$N\tau \geqslant m > (N - r^2)\tau . \tag{9.5c}$$

The inequality (9.5*c*) is particularly interesting when $N\tau - (N - r^2)\tau = 1$, i.e.

$$r = \frac{1}{\sqrt{\tau}} =: \rho. \tag{9.6}$$

In this case we obtain

$$\Sigma_N^\rho = \{x \in \mathbb{I} \mid x \cdot x = N + [N\tau]\tau\} \qquad N = 0, 1, 2, \ldots \tag{9.7}$$

where $[N\tau]$ is the integer part of $N\tau$.

Consider now the inflational symmetry

$$x \mapsto \tau x$$

that maps Σ^ρ bijectively onto $\Sigma^{\rho/\tau} \subset \Sigma^\rho$. We have

$$x \in \Sigma_N^\rho \Rightarrow x \cdot x = N + [N\tau]\tau$$

$$\Rightarrow \tau x \cdot \tau x = N(\tau + 1) + [N\tau](2\tau + 1) = (N + [N\tau]) + (N + 2[N\tau])\tau$$

and hence, in view of (9.7),

$$\tau \Sigma_N^\rho \subset \Sigma_{N+[N\tau]}^\rho \tag{9.8}$$

$$[(N + [N\tau])\tau] = N + 2[N\tau] . \tag{9.9}$$

This brings us to the Lucas sequences

$$\{L_n\} = \{g_1 F_{n-1} + g_2 F_n\}_{n=1}^\infty \tag{9.10}$$

where $\{F_n\}_0^\infty = \{0, 1, 1, 2, 3, 5, \ldots\}$ is the Fibonacci sequence and g_1, g_2 are fixed natural numbers. Thus

$$L_1 = g_2 \qquad L_2 = g_1 + g_2 \qquad \text{and} \qquad L_{n+1} = L_n + L_{n-1} . \tag{9.11}$$

Beginning with $g_2 = N$, $g_1 = [N\tau] - N$, where $N \in \mathbb{Z}_+$, we obtain the Lucas sequence

$$\{L_n(N)\} = \{N,\ [N\tau],\ N + [N\tau],\ N + 2[N\tau], \ldots\}$$

By successive applications of (9.8), (9.9), and (9.7) we obtain for all odd n

$$\tau \Sigma_{L_n(N)}^\rho \subset \Sigma_{L_{n+2}(N)}^\rho, \tag{9.12}$$

$$x \in \Sigma_{L_n(N)}^\rho \Rightarrow x \cdot x = L_n(N) + L_{n+1}(N)\tau, \tag{9.13}$$

$$L_{n+1}(N) = [\tau L_n(N)] . \tag{9.14}$$

Remarkably (9.12) is an equality:

$$\tau \Sigma^{\rho}_{L_n(N)} = \Sigma^{\rho}_{L_{n+2}(N)}. \tag{9.15}$$

Indeed, $x \in \Sigma^{\rho}_{L_{n+2}(N)} \Rightarrow x \cdot x = L_{n+2}(N) + L_{n+3}(N)\tau$, so that

$$\sigma x \cdot \sigma x = (\sigma + 1)(L_{n+2}(N) + \tau L_{n+3}(N)) = L_n(N) + \tau L_{n+1}(N) .$$

In view of (9.14) and (9.7), $\sigma x \in \Sigma^{\rho}_{L_n(N)}$ and so finally $x = -\tau(\sigma x) \in \tau \Sigma^{\rho}_{L_n(N)}$.

We define

$$s_N = \operatorname{card} \Sigma^{\rho}_N \qquad N = 0, 1, 2, \ldots . \tag{9.16}$$

In view of (9.15) we have for all N

$$s_N = s_{N+[N\tau]} . \tag{9.17}$$

We observe from (6.15) and (6.16) that each shell Σ^{ρ}_N has complete H_4-symmetry. In particular since left multiplication by elements of I acts without fixed points, each shell decomposes into I-orbits of 120 elements each. Consequently we have

$$120 \mid s_N \qquad \text{for all } N . \tag{9.18}$$

In [11] an explicit formula for card Σ^{r}_N is given for all $N \in \mathrm{N}$, $r > 0$. In particular for the case $r = \rho$ this for mula becomes

$$s_N = 120 \sum_{\mathbf{b} | (N+[N\tau]\tau)\mathbb{Z}[\tau]} (\mathbb{Z}[\tau] : \mathbf{b}) . \tag{9.19}$$

The sum runs over all right ideals of $\mathbb{I}$ dividing $(N+[N\tau]\tau)\mathbb{Z}[\tau]$ and $(\mathbb{Z}[\tau] : \mathbf{b})$ is the index of $\mathbf{b}$ as a subgroup of $\mathbb{Z}[\tau]$. Although throughout the present paper the ring structure of $\mathbb{I}$ is not really used, the derivation of (9.19) and the more general formulas of [11] depend heavily on the fact that $\mathbb{I}$ is a maximal order in $\mathbb{H}_{\mathbb{F}}$.

Our efforts to prove (9.19) were inspired by the conjectured formula (which is actually incorrect) in [6]. The first place where our formula differs from theirs is at $N = 55$.

10. Concluding remarks

Results of this paper pertain to quasicrystals in dimensions 4, 3, 2, and 1. In the article we have exploited the unique position of the E_8 root lattice in mathematics through its relation with the special set of quaternions, the icosians, and their relation with the quadratic extension $\mathbb{F}$ of the rationals by $\sqrt{5}$. Generalization to other structures is possible but it is neither straightforward nor simple. However, the E_8 playground set up here is large enough that practically all the interesting known quasiperiodic structures in physics, involving five-fold symmetry, can be found in it. In section 8 we have pointed out some of the possibilities.

In this article a quasicrystal is a set of points: projections of lattice points in double dimension. In the literature one is sometimes interested in quasiperiodic tilings. A quasicrystal is then viewed as a set of n-dimensional tiles in n dimensions, the important cases being $n = 2, 3$. For this we need to start from a tiling of the $2n$-dimensional space invariant under the corresponding Coxeter group W, and project appropriate faces of the tiles using $\pi_\perp$ and $\pi_\parallel$ from this paper. An explicit general description of tilings generated by Coxeter groups in Euclidean and other spaces for an arbitrary finite dimension recently became easy [14, 18]. Moreover, the formalism here is ready-made for studying the quasiperiodic tilings in four (Euclidean) dimensions and their subsequent projections to any lower dimension.

The quasicrystals in this article 'live' in Euclidean spaces of dimensions up to 4 where certain unit quaternions (icosians) serve as basis vectors. As was illustrated in section 9, the quaternions not only offer considerable convenience in our construction but are crucial in understanding more subtle aspects of the theory.

Appendix. The icosian ring as a lattice

In this section we provide an explicit description in terms of coordinates of $\mathbb{I}$ as a lattice in $\mathbb{R}^4$.

The icosian ring $\mathbb{I}$ is the additive group generated by the three sets of elements (3.1). The first set of points generates $\mathbb{Z}^4$. Adjoining to it the second set we obtain

$$L_4 := \{(a_1, \ldots, a_4) \mid a_i \in \mathbb{Z} \quad \text{for all } i \quad \text{or} \quad a_i \in \mathbb{Z} + \tfrac{1}{2} \quad \text{for all } i\}$$

the index $[L_4 : \mathbb{Z}^4] = 2$, and $L_4/\mathbb{Z}^4$ is generated by the icosian $\frac{1}{2}(1, 1, 1, 1)$.

Since $\tau\mathbb{I} \subset \mathbb{I}$, we obtain

$$L_4 + \tau L_4 = \{a + \tau b \mid a, b \in L_4\} \subset \mathbb{I}.$$

Now consider $\frac{1}{2}(0, 1, \sigma, \tau) = \frac{1}{2}(0, 1, 1 - \tau, \tau) = \frac{1}{2}(0, 1, 1, 0) + \frac{1}{2}\tau(0, 0, -1, 1)$. Define

$$\hat{L}_4 = \{(a_1, \ldots, a_4) \mid a_i \in \mathbb{Z} \cup (\mathbb{Z} + \tfrac{1}{2}),\ \text{card}\ \{i \mid a_i \in \mathbb{Z} \text{ is even}\}\}.$$

Let us show that $\hat{L}_4/L_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Set

$$a := \tfrac{1}{2}(1, 1, 0, 0) \qquad b := \tfrac{1}{2}(0, 1, 1, 0) \qquad c := \tfrac{1}{2}(1, 0, 1, 0).$$

If $x = (x_1, x_2, x_3, x_4) \in \hat{L}_4 \backslash L_4$ then exactly two of the x_i lie in $\mathbb{Z} + \frac{1}{2}$. Since $\frac{1}{2}(1, 1, 1, 1) \in L_4$,

$$\begin{aligned} a &\equiv -a \equiv -a + \tfrac{1}{2}(1, 1, 1, 1) = \tfrac{1}{2}(0, 0, 1, 1) \bmod L_4 \\ b &\equiv \tfrac{1}{2}(1, 0, 0, 1) \bmod L_4 \\ c &\equiv \tfrac{1}{2}(0, 1, 0, 1) \bmod L_4. \end{aligned}$$

Together with $0 = \frac{1}{2}(0, 0, 0, 0)$ this covers all possibilities for $\hat{L}_4/L_4$. Thus

$$\hat{L}_4 = L_4 + (a + L_4) + (b + L_4) + (c + L_4)$$

and $\hat{L}_4/L_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

Now consider the action of the symmetric group S_4 on $\hat{L}_4$ determined by permuting components. Then L_4 is an S_4 invariant subspace. Hence S_4 acts on $\hat{L}_4/L_4$.

Let $K = \{1, (12)(34), (13)(24), (14)(23)\}$,

$$K \lhd S_4 \qquad S_4/K \simeq S_3.$$

We have

$$(12)(34)a = a$$
$$(13)(24)a = \tfrac{1}{2}(0,0,1,1) \equiv a \bmod L_4$$
$$(12)(34)a = \tfrac{1}{2}(0,0,1,1) \equiv a \bmod L_4\,.$$

The same happens for b, c. Thus K acts trivially on $\hat{L}_4/L_4$ and we get an action of S_3 on $\hat{L}_4/L_4$ from $S_4/K \simeq S_3$.

The two non-trivial elements of A_3 are (123) and (132),

$$(123)a = (123)\tfrac{1}{2}(1,1,0,0) = \tfrac{1}{2}(0,1,1,0) = b$$
$$(123)b = (123)\tfrac{1}{2}(0,1,1,0) = \tfrac{1}{2}(1,0,1,0) = c$$
$$(123)c = a\,.$$

We can now determine $\mathbb{I}$: we know that

$$\hat{L}_4 + \tau\hat{L}_4 \supset \mathbb{I} \supset L_4 + \tau L_4 \qquad \text{and} \qquad [\hat{L}_4 + \tau\hat{L}_4 : L_4 + \tau L_4] = 16\,.$$

We claim that

$$\mathbb{I} = \mathbb{J} := \{x + \tau y \mid x, y \in \hat{L}_4,\ (123)y \equiv x \bmod L_4\}\,.$$

It is immediate from the definition that $\mathbb{J}$ is a group and $L_4 + \tau L_4 \subset \mathbb{J}$. Furthermore, since for $x + y\tau \in \mathbb{J}$ the value of $x \bmod L_4$ is determined by y and $[\hat{L}_4 : L_4] = 4$, we see that $[\mathbb{J} : L_4 + \tau L_4] = 4$. Now the generators of $\mathbb{I}$ all lie in $\mathbb{J}$ and hence $\mathbb{I} \subset \mathbb{J}$. Finally it is clear that $\{y \mid x + y\tau \in \mathbb{I},\ x, y \in \hat{L}_4\}$ is precisely $\hat{L}_4$ and so $\mathbb{I}/(L_4 + \tau L_4) = \mathbb{J}/(L_4 + \tau L_4)$, whence $\mathbb{I} = \mathbb{J}$.

Acknowledgment

The work was supported in part by the Natural Sciences and Engineering Research Council of Canada (RVM and JP), by the Fonds FCAR of Québec, and by the Killam Research Foundation (JP).

References

[1] Guyot P, Kramer P and de Boissieu M 1989 Quasicrystals *Rep. Prog. Phys.* **79** 1373–425

[2] Baake P, Kramer P, Schlottmann M and Zeidler D 1990 Planar patterns with fivefold symmetry as sctions of periodic structures in 4-space *Int. J. Mod. Phys.* B 2217–68

[3] Baake M, Joseph D, Kramer P and Schlottmann M 1990 Root lattices and quasicrystals *J. Phys. A: Math. Gen.* **23** L1037–41

[4] Kramer P, Papadopolos Z and Zeidler D 1992 Concepts of symmetry in quasicrystals: root lattice D_6, icosahedral group, etc ... *Group Theory in Physics* (AIP Conference Proc.) **266** 179–200

[5] Elser V and Sloane N J A 1987 A highly symmetric four-dimensional quasicrystal *J. Phys. A: Math. Gen.* **20** 6161

[6] Sadoc J F and Mosseri R 1992 The E_8 lattice and quasicrystals: geometry, number theory, and quasicrystals *J. Phys. A: Math. Gen.* to be published

[7] Sadoc J F and Mosseri R 1993 The E_8 lattice and quasicrystals *J. Non-crystalline Solids* **153, 154** 247–52

[8] Scherbak O P 1988 Wavefronts and reflection groups *Russ. Math. Surveys* **43: 3** 149–94

[9] Witt E 1941 Spiegelungsgruppen und aufzählung halbeinfacher Liescher ringe *Abh. Math. Sem. Univ. Hamburg* **14** 289–322

[10] Tits J 1980 Quaternions over $Q(\sqrt{5})$, Leech lattice and the sporadic group of Hall–Janko *J. Algebra* **63** 56–75

[11] Moody R V and Weiss A, The shelling of E_8 quasicrystals, in preparation

[12] du Val P 1964 *Homographies, Quaternions and Rotations* (Oxford: Oxford University Press)

[13] Moody R V, Patera J and Rand D 1992 *simpLie (Macintosh computer software v2)* Centre de Recherches Mathématiques, Université de Montréal

[14] Moody R V and Patera J 1992 Voronoi and Delaunay cells of root lattices: Classification of their faces and facets by Coxeter–Dynkin diagrams *J. Phys. A: Math. Gen.* **25** 5089–134

[15] Coxeter H M S and Moser W O J 1965 Generators and relations for discrete groups *Ergebnisse der Mathematik* 2nd edn (New York: Springer)

[16] Deodhar V V 1982 On root systems of a Coxeter group *Commun. Algebra* **10** 611–30

[17] Humphreys J E 1990 *Reflection Groups and Coxeter Groups* (Cambridge: Cambridge University Press)

[18] Moody R V and Patera J 1992 Voronoi domains and dual cells in the generalized kaleidoscope with application to root and weight lattices *Preprint* CRM-1803 pp 1–49